0.

X smooth Fano$/\mathbb{C}$ (i.e. X-proj, $-K_X$ ample)

When is X K-(poly)stable?

$\overset{\text{YTD}}{\Updownarrow}$ ← algebraic condition on the degenerations of X

$\exists$ a Kähler-Einstein metric on X

Also: K-moduli spaces of Fano varieties.

K-polystable varieties of a fixed volume are parametrized by a projective scheme!

Criterion for K-stability:

Tian (1987): α-invariant $n = \dim X$

If $\alpha(X) > \frac{n}{n+1} \Longrightarrow X$ is K-stable

I. Complex α-invariant

KLT singularities: (X, Δ) pair

normal variety $\quad$ ≥ 0 $\mathbb{Q}$-divisor

Assume: rK_X, $r\Delta$ Cartier for some $r>0$.

$\mu: Y \to X$ log resolution of (X, Δ)

and write $\quad K_Y - \mu^* K_X = \sum a_i E_i$

$\mu^* \Delta = \sum b_i E_i$ $\quad a_i, b_i \in \mathbb{Q}$

Then (X, Δ) is KLT $\Longleftrightarrow a_i - b_i > -1 \ \forall i$

Tian, Demailly: X KLT Fano variety

$$\alpha(X) = \sup \left\{ t \geq 0 \;\middle|\; \begin{array}{l} (X, tD) \text{ is KLT} \\ \forall\, D \geq 0 \ \mathbb{Q}\text{-divisor} \\ \quad D \sim_{\mathbb{Q}} -K_X \end{array} \right\}$$

$$= \inf_{n \geq 1} \inf_{D \sim -nK_X} \operatorname{lct}(X, D) \cdot n$$

Fact: $\alpha(X) > 0$. (not clear if X is not smooth).

Eg. $X = \mathbb{P}^n$, $-K_X \sim (n+1)H$

$$\alpha(X) = \frac{1}{n+1}.$$

We would like to find an analog in characteristic $p>0$. For that we will replace KLT with global F-regularity.

II. Global F-regularity:

k - perfect field of char $p>0$.

$$X \xrightarrow{F} X$$

Frobenius map

(X, Δ) pair as before. Then

Def:

(X,Δ) is globally F-regular (GFR) if for any effective divisor $D \geq 0$, $\exists$ some $e \gg 0$ such that

$$\varphi: \mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(\lceil (p^e-1)\Delta \rceil + D)$$

splits $/\mathcal{O}_X$ i.e. $\exists\, \psi$ s.t. $\psi \circ \varphi = id$

Fact: (Schwede-Smith) X Fano, $D \sim_{\mathbb{Q}} -K_X$

For $0 \leq t < 1$

Then (X, tD) is KLT $\iff$ $(X, tD) \bmod p$ is globally F-regular $\forall\, p \gg 0$.

There is a local version of F-regularity that is also related to KLT singularities. But we will focus only on the global one.

III. α_F-invariant

X globally F-regular Fano variety $/k$

$$\alpha_F(X) = \sup\left\{ t \geq 0 \;\middle|\; \begin{array}{l} (X, tD) \text{ is globally F-reg} \\ \forall\, D \geq 0 \ \mathbb{Q}\text{-div},\ D \sim_{\mathbb{Q}} -K_X \end{array} \right\}$$

Remarks: If $D \sim_{\mathbb{Q}} -K_X$, $0 \leq t < 1$

$$(X, tD) \text{ GFR} \overset{\text{Analogy}}{\approx} (X, tD) \text{ is KLT}$$

$$\therefore\ \alpha_F(X) \overset{\text{Analogy}}{\approx} \min\{1, \alpha(X)\}$$

Let $S = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(-mrK_X))$ s.t. $-rK_X$ Cartier

Then, (X, tD) is KLT $\Longleftrightarrow$ $(\mathrm{Spec}(S), t\widetilde{D})$ is KLT.

Fails for local F-regularity

IV. Thm (–). X globally F-reg Fano variety /k.

Then, 1) $\alpha_F(X) > 0$ (also follows from results of Kenta Sato)

2) $\alpha_F(X) \leq 1/2$.

3) $\alpha_F(X) = 1/2 \iff$

$$s(X) = \frac{\mathrm{vol}(-K_X)}{2^d\,(d+1)!}$$

$s(X) :=$ F-signature of X

- another asymptotic invariant of singularities
- has similar properties to local normalized volume

4) If X is toric Fano, then

$$\alpha_F(X) = \alpha_{\mathbb{C}}(X) \leftarrow \text{complex } \alpha\text{-invariant.}$$

- F-signature: (R, m, k) local ring

$$a_e(R) = \max\{a \mid F_*^e R \cong R^{\oplus a} \oplus M \text{ for some } R\text{-module } M\}$$

$$s(R) = \lim_{e\to\infty} \frac{a_e}{p^{e(\dim R)}}$$

X Fano variety, $s(X) = r\, s(S(X, -rK_X))$

s.t. $-rK_X$ is Cartier

- Remark: If X toric Fano variety $/\mathbb{C}$,

then $\alpha(X) \leq 1/2$: well-known.

V. Sketch proof and Example:

1. Lemma:

$$\alpha_F(X) = \sup\left\{ \lambda \;\middle|\; \begin{array}{l} \forall e \geq 1,\ \forall m \leq \lambda(p^e-1), \\ \forall D \in |-mK_X|, \text{ the map} \\ \mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(D) \text{ splits} \end{array} \right\}$$

$$\mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(D)$$

2. Assume X is smooth for simplicity

Main ingredient: duality for Frobenius

$$\mathcal{H}om_{\mathcal{O}_X}(F^e_* \omega_X^{-m}, \mathcal{O}_X) \cong F^e_* \omega_X^{-(p^e-1-m)}$$

for any $m \geq 0$.

In particular, for $0 \leq m \leq p^e - 1$

\# of $\mathcal{O}_X$-summands of

$$F^e_* \omega_X^{-m} = \text{same for } F^e_* \omega_X^{-(p^e-1-m)}$$

3. Suppose $\lambda < \alpha_F(X)$, Then

$\forall\, e \geq 0$, $\forall\, m \leq \lambda(p^e - 1)$.

$$F^e_* \omega_X^{-m} \cong \mathcal{O}_X^{h^0(\omega_X^{-m})} \oplus \mathcal{F} \text{ for some } \mathcal{F}.$$

4. Assume $\alpha_F(X) > 1/2$.

$\rightsquigarrow$ for any $e \gg 0$

$$F^e_* \omega_X^{-0},\ F^e_* \omega_X^{-1},\ \dots,\ F^e_* \omega_X^{-\frac{p^e-1}{2}},\ \dots,\ F^e_* \omega_X^{-(p^e-2)},\ F^e_* \omega_X^{-(p^e-1)}$$

$\mathcal{O}_X$-dual

$\mathcal{O}_X$-duals.

of $\mathcal{O}_X$-summands : Symmetric and reverses

But, $h^0(\omega_X^{-m})$: increasing.

which is a contradiction!

Example: $X = \{x^3+y^3+z^3+w^3=0\} \subseteq \mathbb{P}^3_{\mathbb{Z}}$

$X_p = X \times_{\mathbb{Z}} \mathbb{F}_p$. $X_{\mathbb{C}} = X \times_{\mathbb{Z}} \mathbb{C}$

$$\alpha(X_{\mathbb{C}}) = \frac{2}{3}$$

$$\alpha(X_p) < \frac{1}{2}, \text{ but } \lim_{p\to\infty} \alpha(X_p) = \frac{1}{2}.$$

(follows from Shideler-Tucker, Han-Monsky)

VI. Semicontinuity:

Thm 2: [_] $f: \mathcal{X} \to Y$ flat family of globally F-regular Fano varieties & Y is smooth.

i.e. $-K_{\mathcal{X}/Y}$ is $\mathbb{Q}$-Cartier & f-ample

Then the map

$$Y \ni y \longmapsto \alpha_F(\mathcal{X}_{y^\infty})$$

is lower semicontinuous, where y^∞ denotes the perfect point over $y \in Y$.

Rmk: Corresponding result for α-invariant/$\mathbb{C}$ was proved by Blum-Liu using Nadel vanishing & global generation results.